Backstepping

From simple designs to take-off

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Backstepping

- **Constructive** (=systematic) control design for nonlinear systems
- Applies to systems of **lower triangular** form

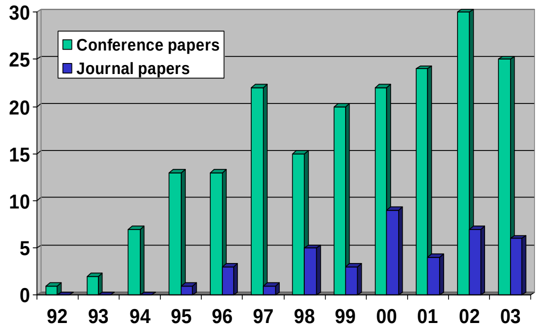
$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_1, x_2, x_3) \\
&\vdots \\
\dot{x}_n &= f_n(x_1, x_2, x_3, \ldots, x_n, u)
\end{aligned}$$

(essentially same as for feedback linearization)

- Can be used to **avoid cancellation** of "useful nonlinearities" (unlike feedback linearization)
- Different flavours: **adaptive**, robust and observer backstepping

Paper statistics

IEEE Explore 1990-2003
Backstepping in title



- Conference papers: 194
- Not adaptive: 108
- Applied 2003: 16 of 25

References

- Books
  - Nonlinear and Adaptive Control Design, 1995 (Krstic, Kanellakopolous, Kokotovic).
  - Constructive Nonlinear Control, 1997 (Sepulchre, Jankovic, Kokotovic).
  - Any recent textbook on nonlinear control.
- Papers
  - The joy of feedback: nonlinear and adaptive, 1991 Bode lecture (Kokotovic).
  - Constructive nonlinear control: a historical perspective, Automatica, 2001 (Kokotovic, Arcak).

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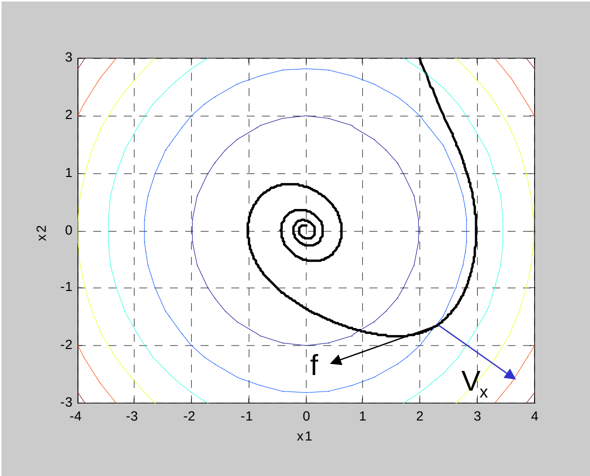
Lyapunov stability (geometric interpretation)



- Dynamics: $\dot{x} = f(x)$
- Lyapunov function: $V(x)$
- For stability: $\dot{V} = V_x f \le 0$

- Dynamics: $\dot{x} = f(x) + g(x)u$
- $V(x)$ is a CLF if

$$\dot{V} = V_x f + V_x g u < 0$$

for some u

Example 1 (backstepping)

$$\begin{aligned} \dot{x}_1 &= x_1^2 + x_2 \\ \dot{x}_2 &= u \end{aligned}$$

Step 1: $x_{2,d} = -x_1^2 - x_1$

$$V_1 = \tfrac{1}{2} x_1^2$$

$$\dot{V}_1 = x_1 \dot{x}_1 = -x_1^2 \le 0$$

if $x_2 = x_{2,d}$

Step 2: $\tilde{x}_2 = x_2 - x_{2,d} = x_2 + x_1^2 + x_1$

$$\begin{cases} \dot{x}_1 = -x_1 + \tilde{x}_2 \\ \dot{\tilde{x}}_2 = u + (2x_1 + 1)(-x_1 + \tilde{x}_2) \end{cases}$$

ϕ

$$V_2 = \tfrac{1}{2} x_1^2 + \tfrac{1}{2} \tilde{x}_2^2$$

$$\dot{V}_2 = x_1(-x_1 + \tilde{x}_2) + \tilde{x}_2(u + \phi)$$

$$= -x_1^2 + \tilde{x}_2(x_1 + u + \phi)$$

$$= -x_1^2 - \tilde{x}_2^2 \le 0$$

if $u = -x_1 - \phi - \tilde{x}_2$

Example 1 (feedback linearization)

$$\dot{x}_1 = x_1^2 + x_2$$
$$\dot{x}_2 = u$$

? Which control law should I choose?

$$y = x_1 = z_1$$
$$\dot{z}_1 = x_1^2 + x_2 = z_2$$
$$\dot{z}_2 = 2z_1z_2 + u$$
$$\Downarrow$$

$u = -2z_1z_2 - k_1z_1 - k_2z_2$ gives stability

! Same control law with $k_1 = k_2 = 2$

"Conclusion"

Backstepping with linearizing virtual control laws and quadratic Lyapunov functions

⟷

Feedback linearization

Easier to tune

Example 1 (backstepping)

$$\dot{x}_1 = x_1^2 + x_2$$
$$\dot{x}_2 = u$$

Step 1:

$$x_{2,d} = -x_1^2 - x_1$$
$$V_1 = \tfrac{1}{2} x_1^2$$
$$\dot{V}_1 = x_1 \dot{x}_1 = -x_1^2 \le 0$$

if $x_2 = x_{2,d}$

Step 2:

$$\tilde{x}_2 = x_2 - x_{2,d} = x_2 + x_1^2 + x_1$$

$$\begin{cases} \dot{x}_1 = -x_1 + \tilde{x}_2 \\ \dot{\tilde{x}}_2 = u + (2x_1 + 1)(-x_1 + \tilde{x}_2) \end{cases}$$

ϕ

$$V_2 = \tfrac{1}{2} x_1^2 + \tfrac{1}{2} \tilde{x}_2^2$$
$$\dot{V}_2 = x_1(-x_1 + \tilde{x}_2) + \tilde{x}_2(u + \phi)$$
$$= -x_1^2 + \tilde{x}_2(x_1 + u + \phi)$$
$$= -x_1^2 - \tilde{x}_2^2 \le 0$$

if $u = -x_1 - \phi - \tilde{x}_2$

Example 2 (adaptive backstepping)

$$\dot{x}_1 = x_1^2 + x_2$$
$$\dot{x}_2 = u + \theta x_2^2$$

Step 1:

$$x_{2,d} = -x_1^2 - x_1$$
$$V_1 = \tfrac{1}{2} x_1^2$$
$$\dot{V}_1 = x_1 \dot{x}_1 = -x_1^2 \le 0$$

Step 2:

$$\tilde{x}_2 = x_2 - x_{2,d} = x_2 + x_1^2 + x_1$$

$$\begin{cases} \dot{x}_1 = -x_1 + \tilde{x}_2 \\ \dot{\tilde{x}}_2 = u + (2x_1 + 1)(-x_1 + \tilde{x}_2) + \theta x_2^2 \end{cases}$$

$$V_2 = \tfrac{1}{2} x_1^2 + \tfrac{1}{2} \tilde{x}_2^2 + \tfrac{1}{2}(\theta - \hat{\theta})^2$$
$$\dot{V}_2 = -x_1^2 + \tilde{x}_2(x_1 + u + \phi + \theta x_2^2) - (\theta - \hat{\theta})\dot{\hat{\theta}}$$

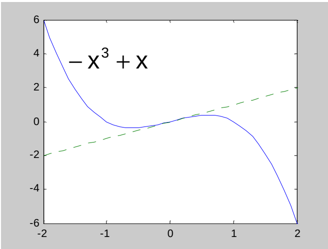
$u = -x_1 - \phi - \tilde{x}_2 - \hat{\theta} x_2^2$ gives

$$\dot{V}_2 = -x_1^2 - \tilde{x}_2^2 + (\theta - \hat{\theta})\left(\tilde{x}_2 x_2^2 - \dot{\hat{\theta}}\right) \le 0$$

if $\dot{\hat{\theta}} = -\tilde{x}_2 x_2^2$

Example 3 (useful nonlinearity)

$$\dot{x}_1 = -x_1^3 + x_1 + x_2$$
$$\dot{x}_2 = u$$



Step 1: $x_{2,d} = -x_1$

$V_1 = W(x_1)$

Step 2: $\tilde{x}_2 = x_2 - x_{2,d} = x_2 + x_1$

$$\begin{cases} \dot{x}_1 = -x_1^3 + \tilde{x}_2 \\ \dot{\tilde{x}}_2 = u - x_1^3 + \tilde{x}_2 \end{cases}$$

$$V_2 = W(x_1) + \tfrac{1}{2}\tilde{x}_2$$

$$\dot{V}_2 = W'(x_1)(-x_1^3 + \tilde{x}_2) + \tilde{x}_2(u - x_1^3 + \tilde{x}_2)$$

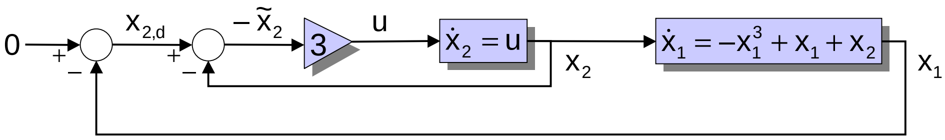
$$= -W'(x_1)x_1^3 + \tilde{x}_2(W'(x_1) + u - x_1^3 + \tilde{x}_2)$$

With $W(x_1) = \tfrac{1}{4}x_1^4$ we can select

$u = -3\tilde{x}_2$ to achieve $\dot{V}_2 = -x_1^6 - 2\tilde{x}_2^2 \le 0$

Example 3 (control law properties)

- Cascaded control implementation



- No cancellations ⇒ gain margin (1/3, ∞)
- Inverse optimal, minimizes

$$\int_0^\infty \left(x_1^6 + \tfrac{1}{2}(x_1 + x_2)^2 + \tfrac{1}{6}u^2\right)dt$$

Design paths

System —backstepping→ CLF, control law

System —backstepping→ CLF —Sontag's formula→ control law (inverse optimal)

Ex 1: $u = -2z_1z_2 - \frac{(z_1 + cz_2)z_1 + \sqrt{(2z_1z_2(z_2 + cz_1) + z_2(z_1 + cz_2))^2 + (z_2 + cz_1)^4}}{cz_1 + z_2}$

System —backstepping→ CLF —H_∞ (Ezal et al. 2000)→ control law (locally optimal, globally inv. opt.)

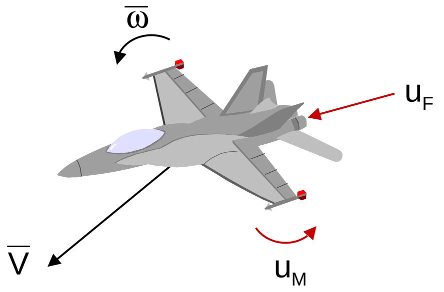
Backstepping control of a rigid body

- **Plug-and-play** flight controller
- Use **vector description** of dynamics for control design

(Extension of CDC 2002 paper by Glad & Härkegård.)

Control problem

- 6 states: V, ω
- 4 inputs: u_F, u_M



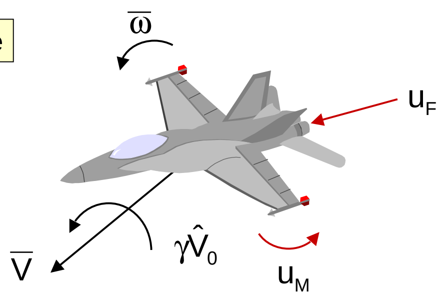
Dynamics

Aerodynamics — Gravity — Engine

$$m\dot{V} = -\omega \times mV + f(V,t) + u_F \hat{n}$$
$$J\dot{\omega} = -\omega \times J\omega + u_M$$

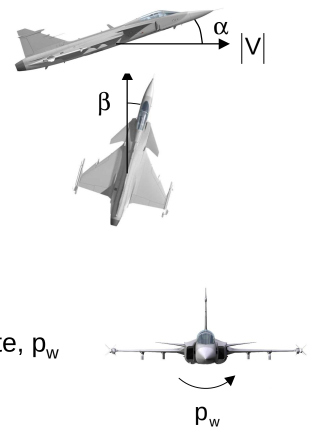
- Stationary motion:

$$V = V_0$$
$$\omega = g(V_0) + \gamma \hat{V}_0$$



Controlled variables

V :
- Angle of attack, α
- Sideslip angle, β
- Total velocity, $|V|$



$\omega^T \hat{V}$:
- Velocity vector roll rate, p_w

Backstepping novelties

$$
\begin{aligned}
m\dot{V} &= -\omega \times mV + f(V,t) + u_F \hat{n} \\
J\dot{\omega} &= -\omega \times J\omega + u_M
\end{aligned}
$$

- No clear lower triangular form
- Vector states (not scalars)
- MIMO problem

Backstepping design

Step 1: $m\dot{V} = -\omega \times mV + f(V,t) + u_F \hat{n}$

$$W_1 = \tfrac{m}{2}(V - V_0)^T (V - V_0)$$

$$u_F = \bar{u}_F + u_F^c$$
$$\omega_d = \bar{\omega} + \omega^c$$

cancel f(V,t)

$\omega = \omega_d$ gives

$$\dot{W}_1 = m\bar{\omega}^T (V \times V_0) + \bar{u}_F (V - V_0)^T \hat{n}$$

$\dot{W}_1 \le 0$ if we select

$$\bar{u}_F = -k_1 (V - V_0)^T \hat{n}$$
$$\bar{\omega} = -K_2 (V \times V_0) + \gamma \hat{V}$$

Step 2: $J\dot{\omega} = -\omega \times J\omega + u_M$

$$\tilde{\omega} = \omega - \omega_d$$

$$W_2 = cW_1 + \tfrac{1}{2}\tilde{\omega}^T \tilde{\omega}$$

$\dot{W}_2 \le 0$ if we select

$$u_M = \omega \times J\omega + J\dot{\omega}_d - K_3 \tilde{\omega}$$

Closed loop dynamics

- **Independent** of nonlinear force f(V,t) (but not linear)
- Good **decoupling**
- **Easy to tune** locally linear dynamics of |V|, α, β and ω
- **Singular** at α = 90 deg
- **Robustness**?

Flight simulation