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Brief paper

Resolving actuator redundancy—optimal control vs. control allocation☆

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Abstract

This paper considers actuator redundancy management for a class of overactuated nonlinear systems. Two tools for distributing the control effort among a redundant set of actuators are optimal control design and control allocation. In this paper, we investigate the relationship between these two design tools when the performance indexes are quadratic in the control input. We show that for a particular class of nonlinear systems, they give exactly the same design freedom in distributing the control effort among the actuators. Linear quadratic optimal control is contained as a special case. A benefit of using a separate control allocator is that actuator constraints can be considered, which is illustrated with a flight control example.

Keywords: Optimal control; Control allocation; Nonlinear systems; Input constraints; Quadratic programming; Flight control

1. Introduction

Actuator or effector redundancy is one issue to be dealt with when designing controllers for overactuated dynamic systems. The terms actuator and effector are not used quite consistently in the literature, but in, for example, flight control one could talk about actuators driving the control surfaces, which are the effectors of the aircraft. In our case there is no need to distinguish between actuators and effectors since we merely assume that redundancy in any of them leads to our mathematical formulation below. A common approach is to use optimal control design (Athans & Falb, 1966; Bryson & Ho, 1975; Lewis & Syrmos, 1995) to shape the closed-loop dynamics as well as the actuator control distribution in one step. For linear systems in particular, methods like linear quadratic (LQ) control (Kwakernaak & Sivan, 1972; Anderson & Moore, 1989; Dorato, Abdallah, & Cerone, 1995) and $\mathscr{H}_\infty$ control (Zhou, Doyle, & Glover, 1996) are readily available.

An alternative is to separate the regulation task from the control distribution task. With this strategy, the control law specifies only the total control effort to be produced. The distribution of control among the actuators is then decided by a separate *control allocation* module. The resulting control configuration is illustrated in Fig. 1. This strategy can be found in several practical applications such as aerospace control (Durham, 1993; Adams, Buffington, & Banda, 1994; Virnig & Bodden, 1994; Shertzer, Zimpfer, & Brown, 2002) and control of marine vehicles (Lindfors, 1993; Sørdalen, 1997; Johansen, Fossen, & Berge, 2004). Similar control distribution concepts can also be found in biomechanical muscle control (Thelen, Anderson, & Delp, 2003) and yaw stability control for cars (Hattori, Koibuchi, & Yokoyama, 2002).

In this paper, we derive some connections between these two strategies, i.e., between (a) using optimal control to decide the control input directly, and (b) using optimal control to decide the total control effort and then using control allocation to compute the control input. We will study the case when the performance indexes used in the optimal control

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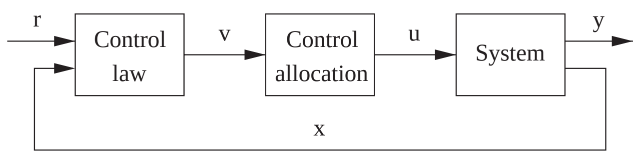

Fig. 1. Control configuration when regulation and control allocation are performed separately.

designs and in the control allocation are quadratic in the control input. This comparison is particularly interesting from a flight control perspective since LQ control design (which is contained as a special case) is a commonly used method today (Gangaas, Bruce, Blight, & Ly, 1986; Stevens & Lewis, 1992; Amato, Mattei, & Scala, 1997; Balas, 2003) and control allocation is possibly becoming one (Hanson, 2002; Balas, 2003). A related comparison between dynamic and static optimization for computing the actuator control distribution can be found in Anderson and Pandy (2001).

The main result to be shown is that for a particular class of overactuated nonlinear systems, the two design strategies offer precisely the same design freedom. Given one design, we show how to select the parameters of the other design to obtain the same control law. We also motivate what benefits a modular design—with a separate control allocator—offers. In particular, actuator constraints can be handled in a potentially better way, which is illustrated with a flight control example.

2. Problem description

Consider a nonlinear system of the form

$$\dot{x} = a(x) + B_u(x)u, \tag{1}$$

where $a(x) \in \mathbb{R}^n$, $B_u(x) \in \mathbb{R}^{n\times m}$, $x(t) \in \mathbb{R}^n$ is the state, and $u(t) \in \mathbb{R}^m$ is the control input. Assume that rank $B_u(x) = k < m\ \forall x$, i.e., that $B_u(x)$ does not have full column rank. This implies that $B_u(x)$ can be factorized as

$$B_u(x) = B_v(x)B(x), \tag{2}$$

where $B_v(x) \in \mathbb{R}^{n\times k}$ and $B(x) \in \mathbb{R}^{k\times m}$ both have rank k. This gives the alternative system description

$$\begin{aligned} \dot{x} &= a(x) + B_v(x)v, \\ v &= B(x)u, \end{aligned} \tag{3}$$

where $v(t) \in \mathbb{R}^k$ can be interpreted as the total control effort produced by the actuators. We will refer to v as the *virtual control input*.

Since $k < m$, B (and also B_u) has a nullspace of dimension $m - k$ in which u can be perturbed without affecting the system dynamics. This means that there are several ways to apportion the control among the actuators, all of which make the system behave the same way. This is the type of actuator redundancy that is typically considered in control allocation applications.

Let us now consider two possible control designs for this class of systems. In Design 1, an optimal control problem is posed directly in terms of u. In Design 2, an optimal control problem is posed in terms of v and the solution is then mapped onto u using control allocation. Quadratic control costs are used in both designs.

Design 1. *Consider the system description* (1). *Determine* $u(t)$ *by solving*

$$\min_{u(\cdot)} \int_0^\infty [q(x) + u^{\mathrm{T}} R_u(x) u]\,\mathrm{d}t, \tag{4}$$

where $q(x) \geqslant 0$ *and* $R_u(x) = R_u(x)^{\mathrm{T}} > 0$.

Design 2. *Consider the system description* (3). *Determine* $v(t)$ *by solving*

$$\min_{v(\cdot)} \int_0^\infty [q(x) + v^{\mathrm{T}} R_v(x) v]\,\mathrm{d}t, \tag{5}$$

where $q(x) \geqslant 0$ *and* $R_v(x) = R_v(x)^{\mathrm{T}} > 0$. *Then determine* $u(t)$ *by solving*

$$\begin{aligned} &\min_{u(t)} \quad u^{\mathrm{T}} W(x) u, \\ &\text{subject to} \quad B(x)u = v, \end{aligned} \tag{6}$$

where $W(x) = W(x)^{\mathrm{T}} > 0$.

We will refer to (6) as l_2-optimal control allocation. It is straightforward to show that (6) has the solution

$$u = W(x)^{-1}B(x)^{\mathrm{T}}(B(x)W(x)^{-1}B(x)^{\mathrm{T}})^{-1}v. \tag{7}$$

Both Designs 1 and 2 involve the solving of optimal control problems. Design 2 can be computed by first calculating $V(x)$ from the Hamilton–Jacobi equation (8) below. The optimal control $v(t)$ is then computed as a state feedback, given by (9).

$$\begin{aligned} 0 = {} & q(x) + V_x(x)a(x) \\ & - \tfrac{1}{4} V_x(x) B_v(x) R_v(x)^{-1} B_v(x)^{\mathrm{T}} V_x(x)^{\mathrm{T}}, \end{aligned} \tag{8}$$

$$v = -\tfrac{1}{2} R_v(x)^{-1} B_v(x)^{\mathrm{T}} V_x(x)^{\mathrm{T}}. \tag{9}$$

For Design 1, R_v, B_v are replaced by R_u, B_u, respectively.

Theorem 3. *Let the Hamilton–Jacobi equation* (8) *have a continuously differentiable solution with* $V(0) = 0$ *in some open set* Ω *containing the origin. Assume that there exists a set* $M \in \Omega$ *such that all solutions of* (3), (9) *starting in* M *remain in* Ω *and converge to the origin. Then the value of the criterion in* (5) *is* $V(x_o)$ *for a trajectory starting in* $x_o \in M$. *The control law* (9) *is optimal in the following sense*: *Let* $\bar{v}$ *be another control that generates a trajectory, starting in* x_o, *remaining in* Ω *and converging to the origin whose value of the criterion in* (4) *is* $\bar{V}$. *Then* $\bar{V} \geqslant V(x_o)$.

Proof. This is a standard result in optimal control. A corresponding global theorem can be found in (Sepulchre, Janković, & Kokotović, 1997, Theorem 3.19) while the restriction to optimality on a set is discussed in (Leitmann, 1981, Chapter 15). □

3. Main result

We will now present the main results of the paper which connect the solutions of Designs 1 and 2.

Theorem 4. *Consider Designs* 1 *and* 2 *and assume that the matrices* R_u *and* R_v *are related as*

$$B(x)R_u(x)^{-1}B(x)^{\mathrm{T}} = R_v(x)^{-1}. \tag{10}$$

Then the following holds.

- *The Hamilton–Jacobi equations associated with Designs* 1 *and* 2 *will be identical. In particular, if one of the problems has a solution satisfying the assumptions of Theorem* 3, *the other one will also and the optimal costs in* (4) *and* (5) *are the same.*
- *If* u^* *and* v^* *are the optimal controls associated with Designs* 1 *and* 2, *respectively, then*

  $$B(x)u^* = v^*$$

  and the corresponding x-trajectories are the same.

Proof. Let u^* and v^* be the optimal controls corresponding to Designs 1 and 2, respectively. Here v^* is computed from (8), (9) while u^* is the solution of

$$0 = q(x) + V_x(x)a(x) - \tfrac{1}{4}V_x(x)B_u(x)R_u(x)^{-1}B_u(x)^{\mathrm{T}}V_x(x)^{\mathrm{T}}, \tag{11}$$

$$u = -\tfrac{1}{2}R_u(x)^{-1}B_u(x)^{\mathrm{T}}V_x(x)^{\mathrm{T}}. \tag{12}$$

Substituting $B_u = B_v B$ and $BR_u^{-1}B^{\mathrm{T}} = R_v^{-1}$ into (11) shows that (11) and (8) are in fact identical. Since

$$Bu^* = -\tfrac{1}{2}BR_u^{-1}B^{\mathrm{T}}B_v^{\mathrm{T}}V_x^{\mathrm{T}} = -\tfrac{1}{2}R_v^{-1}B_v^{\mathrm{T}}V_x^{\mathrm{T}} = v^*$$

implies $B_u u^* = B_v v^*$, the optimal trajectories will be the same. □

This theorem does not answer the question whether u defined by (6) is equal to the optimal u of Design 1. That is the subject of the next theorem.

Theorem 5. *The control laws generated by Designs* 1 *and* 2 *are the same in the following two cases.*

- *If, for given* R_u, *the matrices* R_v *and W are chosen as*

  $$R_v(x) = [B(x)R_u(x)^{-1}B(x)^{\mathrm{T}}]^{-1}, \quad W(x) = R_u(x). \tag{13}$$

- *If, for given* R_v *and W, the matrix* R_u *is chosen as*

  $$R_u(x) = W(x) + B(x)^{\mathrm{T}}[R_v(x) - (B(x)W(x)^{-1}B(x)^{\mathrm{T}})^{-1}]B(x). \tag{14}$$

Proof. According to Theorem 4 the virtual controls generated by the two designs as well as the associated optimal costs are equal if (10) holds. Comparing (7), (9) with (12) we see that the actual control laws generated are the same if

$$R_u^{-1}B^{\mathrm{T}} = W^{-1}B^{\mathrm{T}}(BW^{-1}B^{\mathrm{T}})^{-1}R_v^{-1} \tag{15}$$

which is a stronger condition than (10). For given R_u, selecting R_v and W as in (13) clearly satisfies this relation. To derive (14) we rewrite (15) as

$$X^{-1}A^{\mathrm{T}} = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}R_v^{-1} = A^{\dagger}R_v^{-1},$$

where $X = W^{-1/2}R_uW^{-1/2}$, $A = BW^{-1/2}$, $W^{1/2}$ is the symmetric square root of W, and the pseudoinverse $A^{\dagger}$ satisfies $AA^{\dagger} = (A^{\dagger})^{\mathrm{T}}A^{\mathrm{T}} = I$. One solution is given by

$$X^{-1} = A^{\dagger}R_v^{-1}(A^{\dagger})^{\mathrm{T}} + I - A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}A$$

which according to Lemma 6 (see Appendix A) is positive definite and has the inverse

$$X = A^{\mathrm{T}}R_vA + I - A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}A.$$

In the original variables we get the sought expression

$$R_u = W^{1/2}XW^{1/2} = W + B^{\mathrm{T}}[R_v - (BW^{-1}B^{\mathrm{T}})^{-1}]B. \qquad \square$$

3.1. A simple example

Consider the system

$$\dot{x}_1 = x_2, \tag{16}$$

$$\dot{x}_2 = -x_1^3 + u_1 + 2u_2 \tag{17}$$

with the optimization criterion

$$\int_0^{\infty}\left(x_1^4 + \frac{1}{2}x_1^2 + x_2^2 + u_1^2 + 4u_2^2\right)\mathrm{d}t. \tag{18}$$

Here, we have

$$B_u = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B_v}\underbrace{[1 \;\; 2]}_{B} \tag{19}$$

so the Design 2 problem has the system

$$\dot{x}_1 = x_2, \tag{20}$$

$$\dot{x}_2 = -x_1^3 + v, \tag{21}$$

with criterion

$$\int_0^{\infty}\left(x_1^4 + \frac{1}{2}x_1^2 + x_2^2 + \frac{1}{2}v^2\right)\mathrm{d}t, \tag{22}$$

where we have used

$$R_v^{-1} = BR_u^{-1}B^{\mathrm{T}} = [1 \;\; 2]\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2.$$

The Hamilton–Jacobi equation becomes

$$0 = x_1^4 + \frac{1}{2}x_1^2 + x_2^2 + x_2\frac{\partial V}{\partial x_1} - x_1^3\frac{\partial V}{\partial x_2} - \frac{1}{2}\left(\frac{\partial V}{\partial x_2}\right)^2$$

and has the solution

$$V = \tfrac{1}{2}x_1^4 + x_1^2 + x_1x_2 + x_2^2$$

with control law

$$v = -\frac{\partial V}{\partial x_2} = -x_1 - 2x_2.$$

The optimal control in terms of u_1 and u_2 can now be calculated from

$$\begin{aligned} &\min_u \quad (u_1^2 + 4u_2^2),\\ &\text{subject to} \quad u_1 + 2u_2 = -x_1 - 2x_2 \end{aligned} \tag{23}$$

with the result $u_1 = -\frac{1}{2}x_1 - x_2$, $u_2 = -\frac{1}{4}x_1 - \frac{1}{2}x_2$.

However, it is also possible to change the weighting of u_1 and u_2 by solving

$$\begin{aligned} &\min_u \quad (\alpha u_1^2 + \beta u_2^2),\\ &\text{subject to} \quad u_1 + 2u_2 = -x_1 - 2x_2 \end{aligned} \tag{24}$$

for arbitrary positive α and β. The solution is then

$$u_1 = -\frac{\beta}{4\alpha + \beta}(x_1 + 2x_2), \quad u_2 = -\frac{2\alpha}{4\alpha + \beta}(x_1 + 2x_2)$$

showing that it is possible to shift the control effort between u_1 and u_2. According to Theorem 5 all these choices are optimal, corresponding to

$$R_u = \begin{bmatrix} \alpha + f(\alpha,\beta) & 2f(\alpha,\beta) \\ 2f(\alpha,\beta) & \beta + 4f(\alpha,\beta) \end{bmatrix},$$

$$f(\alpha,\beta) = \frac{\beta + 4\alpha - 2\alpha\beta}{2\beta + 8\alpha}$$

and the same optimal cost $V(x)$. If α, β are chosen so that $\alpha^{-1} + 4\beta^{-1} = 2$ there is the simple relation $R_u = W$.

4. The linear quadratic case

An important special case is when system (1) is linear and the performance indexes are quadratic also in the state. The optimal control problems (4) and (5) then reduce to standard linear quadratic regulation (LQR) problems. This occurs when $a(x) = Ax$, $q(x) = x^{\mathrm{T}}Qx$ where $Q \geqslant 0$, and the matrices B_u, B_v, B, R_u, R_v, and W are constant.

In this case, the Hamilton–Jacobi equation (8) has the solution $V(x) = x^{\mathrm{T}}Px$ where P solves the algebraic Riccati equation

$$0 = Q + A^{\mathrm{T}}P + PA - PB_vR_v^{-1}B_v^{\mathrm{T}}P. \tag{25}$$

If the pair (A, B_v) (or equivalently the pair (A, B_u)) is stabilizable and the pair (A, Q) is detectable, then (25) has a unique positive definite solution P such that the optimal control

$$v = -R_v^{-1}B_v^{\mathrm{T}}Px$$

is asymptotically stabilizing (Kwakernaak & Sivan, 1972; Anderson & Moore, 1989; Dorato et al., 1995). Hence, these conditions certify that the assumptions in Theorem 3 hold globally so that the conversion rules of Theorem 5 can be applied to go from a unified Design 1 to a modular Design 2 or vice versa.

5. Discussion

Let us now discuss the implications of Theorems 4 and 5, relating optimal control design to l_2-optimal control allocation.

The main message is that the two approaches described in Section 2 give the designer exactly the same freedom to shape the closed-loop dynamics and to distribute the control effort among the actuators. Given the design parameters of one design, Theorem 5 states how the parameters of the other design should be selected to achieve precisely the same control law.

So why then split the control design into two separate tasks? Let us list some benefits of using a modular control design.

- *Solving the Hamilton–Jacobi equation.* In the nonlinear case, the solution of the Hamilton–Jacobi equation (8) usually has to be done numerically and can involve heavy computations. However, if R_v is kept constant and W is varied it is not necessary to recompute the solution of (8), but it is still possible to change the weighting of the different components of u. The second part of Theorem 5 shows that this can be done without losing optimality with respect to the criterion in Design 1. It is then possible to view W in (14) as a parameterization of matrices R_u that have the same Hamilton–Jacobi equation.
- *Facilitates tuning.* In Design 1, modifying an element of the control input weighting matrix, R_u, will affect the control distribution as well as the closed-loop behavior of the system. In Design 2, the tuning of the closed loop dynamics is separated from the design of the control distribution.
- *Easy to reconfigure.* An actuator failure can sometimes be modeled as a change in the B-matrix. In Design 2, this only affects the control allocation. Hence, if the failure is detected, the new B-matrix can be used for control allocation, while the original virtual control law can be kept, provided that the damaged system can still be controlled.
- *Arbitrary control allocation method.* The condition in Theorem 4 for the two approaches to give the same x-trajectories does not involve the control allocator.

Hence, if we select R_v as in (10), we can choose any control allocation mapping $u = h(v)$ in Design 2 such that $Bh(v) = v$, without altering the closed loop dynamics from Design 1. For a survey of control allocation methods, see, e.g., Bordignon (1996), Bodson (2002), Härkegård (2003).

- *Actuator constraints.* With a separate control allocator, actuator constraints can be handled to some extent. If the control input is bounded by $\underline{u} \leqslant u(t) \leqslant \overline{u}$, the control allocation problem in Design 2 can be reformulated as

$$u = \arg\min_{u \in \mathscr{U}} u^{\mathrm{T}} W u,$$
$$\mathscr{U} = \arg \min_{\underline{u} \leqslant u \leqslant \overline{u}} (Bu - v)^{\mathrm{T}} W_v (Bu - v). \tag{26}$$

Given $\mathscr{U}$, the set of feasible control inputs that minimize $\|Bu - v\|_{W_v}$, we pick the control input that minimizes $\|u\|_W$. In this way, the control capabilities of the actuator suite can be fully exploited before the closed loop performance is degraded. Also, when $Bu = v$ is not attainable due to the constraints, W_v allows the designer to prioritize between the components of the virtual control input. The optimization problem (26) can be efficiently solved using, e.g., active set methods (Härkegård, 2002) or interior point methods (Petersen & Bodson, 2003).

Remark. It should be stressed that including the constraints in the control allocation is not equivalent to the more complex problem of including the constraints in the optimal control problem in Design 1. This requires the constraints to be considered for all future times. With constrained control allocation the constraints are only considered pointwise in time which can be viewed as a "poor man's constrained optimal control strategy".

Apparently, a modular design has several potential benefits. Unfortunately, not all systems with more actuators than controlled variables display the type of redundancy that can be resolved using control allocation. In some cases however, proper model approximations can be made to achieve modularity, as we will see in the design example in the following section.

6. Flight control example

To investigate the potential benefits of a modular optimal control design we use a flight control example based on the ADMIRE model (ADMIRE ver. 3.4h, 2003; Backström, 1997). ADMIRE describes a small single engine fighter with a delta-canard configuration. To induce actuator saturations, we consider a low-speed flight case, Mach 0.22, altitude 3000 m, where the control surface efficiency is poor.

6.1. Aircraft model

The linearized aircraft model is given by

$$x = [\alpha \;\; \beta \;\; p \;\; q \;\; r]^{\mathrm{T}} - x_{\mathrm{lin}},$$
$$y = [\alpha \;\; \beta \;\; p]^{\mathrm{T}} - y_{\mathrm{lin}},$$
$$\delta = [\delta_c \;\; \delta_{re} \;\; \delta_{le} \;\; \delta_r]^{\mathrm{T}} - \delta_{\mathrm{lin}},$$
$$u = [u_c \;\; u_{re} \;\; u_{le} \;\; u_r]^{\mathrm{T}} - u_{\mathrm{lin}},$$
$$\begin{bmatrix} \dot{x} \\ \dot{\delta} \end{bmatrix} = \begin{bmatrix} A & B_x \\ 0 & -B_\delta \end{bmatrix} \begin{bmatrix} x \\ \delta \end{bmatrix} + \begin{bmatrix} 0 \\ B_\delta \end{bmatrix} u, \tag{27}$$

where α = angle of attack, β = sideslip angle, p = roll rate, q = pitch rate, and r = yaw rate are the aircraft state variables (see, e.g., Stevens & Lewis (1992)), δ and u contain the actual and the commanded deflections of the canard wings, the right and left elevons, and the rudder, respectively, and x_{lin}, y_{lin}, etc. are the points of linearization. The control surfaces are limited by

$$\delta_c \in [-55, 25] \cdot \frac{\pi}{180}, \quad \delta_{re}, \delta_{le}, \delta_r \in [-30, 30] \cdot \frac{\pi}{180}$$

and have first-order dynamics with a time constant of 0.05 s corresponding to $B_\delta = 20I$. For the considered flight case,

$$A = \begin{bmatrix} -0.5432 & 0.0137 & 0 & 0.9778 & 0 \\ 0 & -0.1179 & 0.2215 & 0 & -0.9661 \\ 0 & -10.5128 & -0.9967 & 0 & 0.6176 \\ 2.6221 & -0.0030 & 0 & -0.5057 & 0 \\ 0 & 0.7075 & -0.0939 & 0 & -0.2127 \end{bmatrix},$$

$$B_x = \begin{bmatrix} 0.0069 & -0.0866 & -0.0866 & 0.0004 \\ 0 & 0.0119 & -0.0119 & 0.0287 \\ 0 & -4.2423 & 4.2423 & 1.4871 \\ 1.6532 & -1.2735 & -1.2735 & 0.0024 \\ 0 & -0.2805 & 0.2805 & -0.8823 \end{bmatrix}.$$

For this system, rank $B_\delta = k = m = 4$. Hence, although the number of effectors exceeds the number of controlled variables ($\dim y = 3$), the redundancy is not in a form that can be exploited using control allocation. Let us therefore make the two following approximations:

- The actuator dynamics are neglected, i.e., $\delta = u$ is used.
- The control surfaces are viewed as pure moment generators and their influence on $\dot{\alpha}$ and $\dot{\beta}$ is neglected. This corresponds to zeroing the two top rows of B_x.

This gives the approximate model

$$\dot{x} = Ax + B_u u = Ax + B_v v,$$
$$v = Bu, \tag{28}$$

where

$$B_u = B_v B, \quad B_v = \begin{bmatrix} 0_{2\times 3} \\ I_{3\times 3} \end{bmatrix}$$

and B contains the last three rows of B_x which means $k = 3$. The resulting virtual control input, $v = Bu$, contains the angular accelerations in roll, pitch, and yaw produced by the control surfaces.

6.2. Control design

Let us investigate two different control strategies:

(1) Standard LQ design for the approximate model (28) with weighting matrices Q, R_u.
(2) LQ design and l_2-optimal control allocation for the approximate model (28) with weighting matrices Q, R_v, and W. To handle actuator position constraints, the extended control allocation formulation (26) is used.

The weighting matrices Q and R_u are selected as

$$Q = \mathrm{diag}(10, 10, 2, 1, 10),$$
$$R_u = \mathrm{diag}(10, 10, 10, 10)$$

to achieve desirable characteristics of the short period mode of the aircraft, the dutch roll mode, and the roll mode. For the second design, R_v and W are selected according to (13). The matrix W_v in (26) is selected as

$$W_v = \mathrm{diag}(1, 1, 20)$$

to prioritize the yawing moment in order to maintain a low sideslip angle, β.

To achieve set-point regulation around $y = y_{\mathrm{ref}}$ rather than around $x = 0$, the LQ control laws for u and v are augmented with a feedforward term from the reference y_{ref}, see Härkegård (2003), Chapter 10.

6.3. Simulation results

Fig. 2 shows the simulation results for the two control designs applied to the original linear model (27). Prior to $t = 3$ s, no actuator saturation occurs. In this time interval, Designs 1 and 2 above produce exactly the same control signals in accordance with Theorem 5. When the roll command is applied at $t = 3$ s, the left elevons saturate. In Design 1, this causes an overshoot in the pitch variables α and q. In Design 2, the control allocator copes with the saturation by redistributing as much of the lost control effect as possible to the right elevons and to the canards. The result is that the nominal trajectory, without actuator constraints, is almost completely recovered. Further simulation results can be found in Härkegård (2003).

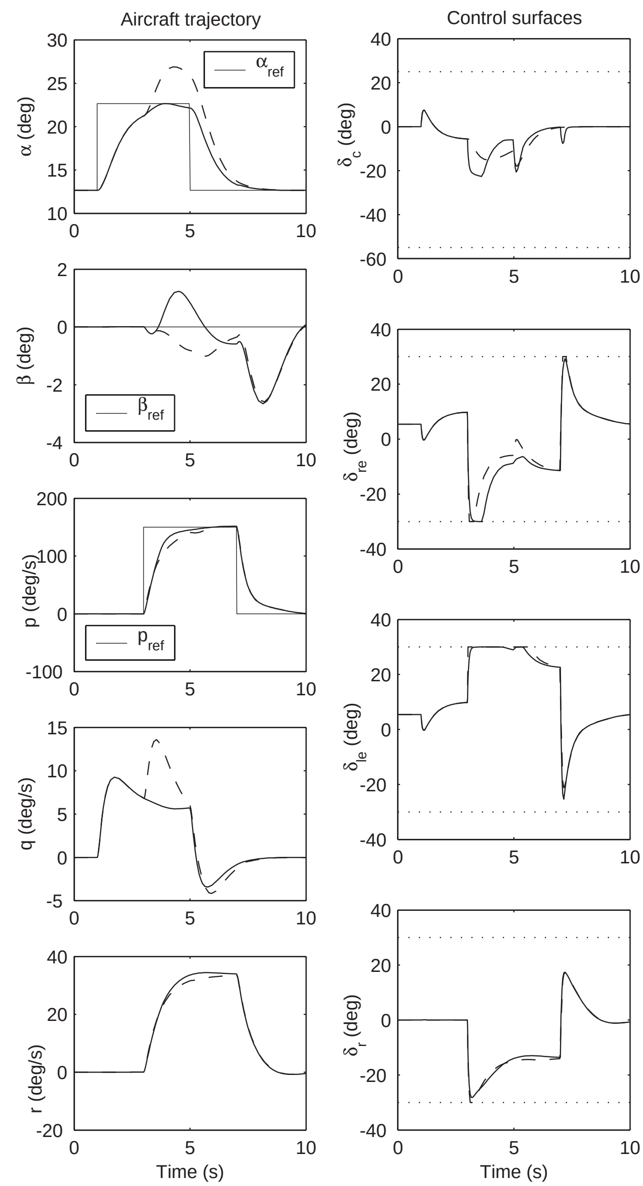

Fig. 2. Aircraft trajectory x (left) and control surface positions δ (right) for control Design 1 (dashed) and Design 2 (solid). In Design 2, the control allocator redistributes the control effort when δ_{le} saturates, thereby preventing an overshoot in the angle of attack, α.

7. Conclusions

In this paper, we have considered optimal control of a class of overactuated nonlinear systems that are affine in the control input. The main result is that when the performance indexes are quadratic in the control, full order optimal control design and reduced order optimal control design in combination with l_2-optimal control allocation offer exactly the same design freedom in shaping the closed loop response and distributing the control effort among the actuators. An important special case is linear quadratic control design.

Theoretically, this is an interesting result in itself since it ties together two useful tools for resolving actuator redundancy. There are also practical implications. Given a full order optimal control design, we have shown how to split this into a new optimal control design with fewer inputs, governing the closed loop dynamics, and a control allocator, distributing the control effort among the actuators. One of the benefits with a separate control allocator is that actuator constraints can be considered, so that when one actuator saturates, the remaining actuators can be used to make up for the loss of control effort, if possible.

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Appendix A. A matrix lemma

Lemma 6. *Let* $A \in \mathbb{R}^{k\times m}$ *have full row rank and let* $R \in \mathbb{R}^{k\times k}$ *be symmetric and positive definite. Then*

$$X = A^{\mathrm{T}}RA + I - A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}A \tag{A.1}$$

is positive definite and has the inverse

$$X^{-1} = A^{\dagger}R^{-1}(A^{\dagger})^{\mathrm{T}} + I - A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}A. \tag{A.2}$$

Proof. Consider the matrix

$$S = \begin{bmatrix} I + A^{\mathrm{T}}RA & A^{\mathrm{T}} \\ A & AA^{\mathrm{T}} \end{bmatrix}.$$

Since $I + A^{\mathrm{T}}RA$ and AA^{T} are both positive definite it holds that (Söderström & Stoica, 1989, Lemma A.3)

$$S > 0 \Leftrightarrow X > 0 \Leftrightarrow Y > 0$$

with X given by (A.1) and with

$$\begin{aligned} Y &= AA^{\mathrm{T}} - A(I + A^{\mathrm{T}}RA)^{-1}A^{\mathrm{T}} \\ &= AA^{\mathrm{T}}(R^{-1} + AA^{\mathrm{T}})^{-1}AA^{\mathrm{T}} > 0, \end{aligned}$$

where we have used the matrix inversion formula (Zhang, 1999, p. 43)

$$(I + CBD)^{-1} = I - C(B^{-1} + DC)^{-1}D. \tag{A.3}$$

Hence, X is positive definite. The expression for X^{-1} follows from (A.3) with $B = I$, $C = A^{\mathrm{T}}(R - (AA^{\mathrm{T}})^{-1})$, and $D = A$. □

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