RESOLVING ACTUATOR REDUNDANCY – CONTROL ALLOCATION vs. LINEAR QUADRATIC CONTROL

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Abstract

When designing control laws for systems with more inputs than controlled variables, one issue to consider is how to deal with actuator redundancy. Two tools for distributing the control effort among a redundant set of actuators are control allocation and linear quadratic control design. In this paper, we investigate the relationship between these two design tools when a quadratic performance index is used for control allocation. We show that for a particular class of linear systems, they give exactly the same design freedom in distributing the control effort among the actuators. The main benefit of using a separate control allocator is that actuator constraints can be considered, which is illustrated with a flight control example.

1 Introduction

Actuator redundancy is one issue to be dealt with when designing controllers for systems with more inputs than outputs. A common approach is to use some optimal control design method, like linear quadratic (LQ) control [3], to shape the closed loop dynamics as well as the actuator control distribution in one step.

An alternative is to separate the regulation task from the control distribution task. With this strategy, the control law specifies only which total control effort should be produced. The distribution of control among the actuators is then decided by a separate *control allocation* module, see Figure 1. This strategy can be found in several practical applications such as aerospace control [6, 10, 9] and control of marine vehicles [11].

In this paper, we derive some connections between these two strategies when quadratic performance indices are used both for control law design and for control allocation. Hence, LQ control and l_2-optimal control allocation will be used to design the control system building blocks in Figure 1. This comparison is particularly interesting from a flight control perspective since LQ design today is a commonly used method [2, 12], and control allocation is possibly becoming one.

The main result to be shown is that for a particular class of overactuated linear systems, the two design strategies offer precisely the same design freedom. Given one design, we show how to select the parameters of the other design to obtain the same control law. We also motivate what benefits a modular

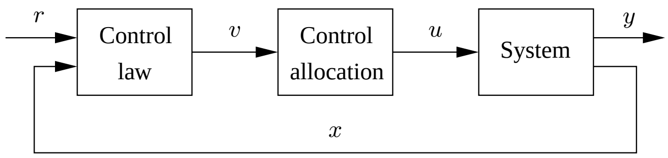

Figure 1: Control system structure when control allocation is performed separately.

design—with a separate control allocator—offers. In particular, actuator constraints can be handled in a potentially better way.

In Section 2, the class of systems considered is introduced. Two different control designs are proposed in Section 3 and are shown to be equivalent in Section 4. Practical implications of this result are discussed in Section 5. Section 6 contains a flight control example and conclusions are drawn in Section 7.

2 System Description

We will consider linear systems of the form

$$\begin{aligned} \dot{x} &= Ax + B_u u \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the system output to be controlled, and (A, B_u) is stabilizable. We assume x to be measured so that full state information is available.

Assume now that $\operatorname{rank}(B_u) = k < m$. This implies that B_u can be factorized as

$$B_u = B_v B$$

where $B_v \in \mathbb{R}^{n \times k}$ and $B \in \mathbb{R}^{k \times m}$. With this, an alternative system description is given by

$$\begin{aligned} \dot{x} &= Ax + B_v v \\ v &= Bu \\ y &= Cx \end{aligned} \tag{2}$$

where $v \in \mathbb{R}^k$ can be interpreted as the total control effort produced by the actuators. We will refer to v as the *virtual control input*.

Since $k < m$, B (and also B_u) has a null space of dimension $m - k$ in which u can be perturbed without affecting the system dynamics. This is the type of actuator redundancy that is typically considered in control allocation applications.

For simplicity, we will restrict ourselves to the case $k = p$, i.e., when the number of virtual control inputs equals the number of variables to be controlled.

3 Two Control Designs

Let the control objective be for the output y to track a constant reference signal r, so that $y = r$ is achieved asymptotically. Based on the two equivalent system descriptions (1) and (2), two different control designs come naturally. We can either consider (1) and design a control law in terms of u directly, or we can consider (2) and first design a control law in terms of v, and then map this onto u. These design alternatives are the topics of Section 3.1 and Section 3.2, respectively.

In developing the control laws, the following lemma is useful.

Lemma 1. *The least squares problem*

$$\begin{aligned} \min_{x} \quad & \|Wx\|^2 = x^T W^T W x \\ \text{subject to} \quad & Ax = y \end{aligned}$$

where W is nonsingular and A has full row rank, is solved by

$$x = W^{-1}(AW^{-1})^{\dagger} y$$

where $A^{\dagger} = A^T(AA^T)^{-1}$ is the pseudoinverse of A.

Proof. See, e.g., [9]. □

3.1 Standard LQ Design

Design 1. *Consider the system description (1) and determine the control input $u(t)$ by solving*

$$\min_{u} \int_0^{\infty} \big((x - x^*)^T Q_1 (x - x^*) + (u - u^*)^T R_1 (u - u^*)\big) dt$$

where $Q_1 = Q_1^T$ is positive semidefinite, $R_1 = R_1^T$ is positive definite, (A, Q_1) is detectable, and x^*, u^* solve

$$\begin{aligned} \min_{x,u} \quad & u^T R_1 u \\ \text{subject to} \quad & Ax + B_u u = 0 \\ & Cx = r \end{aligned} \tag{3}$$

The interpretation of (3) is that if there are several choices of u that achieve $\dot{x} = 0$ and $y = r$, we pick u such that $u^T R_1 u$ is minimized at steady state. The optimal control law is given by the following theorem, based on [7, Thm. 9.2].

Theorem 1. *The optimal control law for Design 1 is given by*

$$\begin{aligned} u(t) &= L_r r - Lx(t) \\ L_r &= R_1^{-\frac{1}{2}}(G_0 R_1^{-\frac{1}{2}})^{\dagger} \\ L &= R_1^{-1} B_u^T S_1 \end{aligned} \tag{4}$$

where

$$G_0 = C(B_u L - A)^{-1} B_u$$

and S_1 is the unique positive semidefinite and symmetric solution to

$$A^T S_1 + S_1 A + Q_1 - S_1 B_u R_1^{-1} B_u^T S_1 = 0$$

Proof. Introduce the residual variables $\tilde{x} = x - x^*$, $\tilde{u} = u - u^*$. The dynamics of $\tilde{x}$ are given by

$$\dot{\tilde{x}} = \dot{x} = Ax + B_u u = A\tilde{x} + B_u \tilde{u}$$

where the last step follows from $Ax^* + B_u u^* = 0$. Standard results from linear quadratic control theory (see, e.g., [3, p. 52]) gives the control law

$$\begin{aligned} \tilde{u} &= -L\tilde{x} \\ L &= R_1^{-1} B_u^T S_1 \end{aligned}$$

where S_1 is the unique positive semidefinite and symmetric solution to the algebraic Riccati equation

$$A^T S_1 + S_1 A + Q_1 - S_1 B_u R_1^{-1} B_u^T S_1 = 0$$

In the original variables we get

$$u = u^* + Lx^* - Lx = u_r - Lx$$

Inserting this into (3) gives us

$$\begin{aligned} \min_{x,u_r} \quad & (u_r - Lx)^T R_1 (u_r - Lx) \\ \text{subject to} \quad & Ax + B_u(u_r - Lx) = 0 \\ & Cx = r \end{aligned}$$

Since (A, B_u) is stabilizable, $A - B_u L$ becomes a Hurwitz matrix and can thus be inverted. Using $B_u = B_v B$ and introducing $v_r = B u_r$, the equality constraints become

$$\begin{aligned} x &= (B_u L - A)^{-1} B_u u_r = (B_u L - A)^{-1} B_v v_r \\ Cx &= C(B_u L - A)^{-1} B_u u_r = C(B_u L - A)^{-1} B_v v_r = r \end{aligned}$$

Assuming that $C(B_u L - A)^{-1} B_v$ ($p \times p$) is nonsingular (or the control problem would not be feasible), we see that v_r, and consequently also x, is completely determined by r. This implies that the objective function can be rearranged as

$$(u_r - Lx)^T R_1 (u_r - Lx) = u_r^T R_1 u_r + f(r)$$

since the mixed term becomes

$$-2x^T L^T R_1 u_r = -2x^T S_1 B_u u_r = -2x^T S_1 B_v v_r$$

and x and v_r are uniquely determined by r. Hence the optimization problem can be restated as

$$\begin{aligned} \min_{u_r} \quad & u_r^T R_1 u_r \\ \text{subject to} \quad & G_0 u_r = r \end{aligned}$$

where $G_0 = C(B_u L - A)^{-1} B_u$, which has the solution

$$u_r = R_1^{-\frac{1}{2}}(G_0 R_1^{-\frac{1}{2}})^{\dagger} r = L_r r$$

according to Lemma 1. □

3.2 LQ Design and Control Allocation

Design 2. *Consider the system description (2) and determine the virtual control input $v(t)$ by solving*

$$\min_v \int_0^\infty \big((x-x^*)^T Q_2 (x-x^*) + (v-v^*)^T R_2 (v-v^*)\big) dt$$

where $Q_2 = Q_2^T$ is positive semidefinite, $R_2 = R_2^T$ is positive definite, (A, Q_2) is detectable, and x^*, v^* solve

$$\begin{aligned} Ax + B_v v &= 0 \\ Cx &= r \end{aligned} \tag{5}$$

Then determine the control input $u(t)$ by solving

$$\begin{aligned} &\min_u \quad \|Wu\| \\ &\text{subject to} \quad Bu = v \end{aligned}$$

where $W = W^T$ is non-singular.

In this case there is no need to minimize $v^T R_2 v$ at steady state since (5) has a unique solution due to that $\dim v = \dim y$.

Theorem 2. *The optimal control law for Design 2 is given by*

$$\begin{aligned} u(t) &= Pv(t) \\ P &= W^{-1}(BW^{-1})^\dagger \end{aligned}$$

Further, the optimal virtual control input is given by

$$\begin{aligned} v(t) &= L_r r - Lx(t) \\ L_r &= G_0^{-1} \\ L &= R_2^{-1} B_v^T S_2 \end{aligned}$$

where

$$G_0 = C(B_v L - A)^{-1} B_v$$

and S_2 is the unique positive semidefinite and symmetric solution to

$$A^T S_2 + S_2 A + Q_2 - S_2 B_v R_2^{-1} B_v^T S_2 = 0$$

Proof. The expressions for P and L follow directly from Lemma 1 and Theorem 1, respectively. Further, solving

$$\begin{aligned} Ax + B_v v &= 0 \\ Cx &= r \\ v &= L_r r - Lx \end{aligned}$$

gives $L_r = \Big(C(B_v L - A)^{-1} B_v\Big)^{-1}$. □

4 Main Result

We will now present the main result of the paper which connects Design 1 and Design 2 in terms of the resulting control input. In the presentation, subscripts 1 and 2 are used to specify which design a certain entity (u, v, etc.) is related to.

Theorem 3. *Consider Design 1 and Design 2. Given Q_1 and R_1, selecting*

$$\begin{aligned} Q_2 &= Q_1 \\ R_2 &= \big(BR_1^{-1}B^T\big)^{-1} \\ W &= R_1^{\frac{1}{2}} \end{aligned} \tag{6}$$

achieves $u_2(t) = u_1(t)$. Conversely, given Q_2, R_2, and W, selecting

$$\begin{aligned} Q_1 &= Q_2 \\ R_1 &= W^2 + B^T\big(R_2 - (BW^{-2}B^T)^{-1}\big)B \end{aligned} \tag{7}$$

achieves $u_1(t) = u_2(t)$.

Proof. We will first consider the case $r = 0$. At the end of the proof we will show that the resulting parameter selection rules lead to $u_1(t) = u_2(t)$ also when $r \neq 0$.

For the control signals to be equal, the virtual control signals must be equal. From Theorem 1 and Theorem 2 we get

$$\begin{aligned} u_1(t) &= -L_1 x(t) = -R_1^{-1} B_u^T S_1 x(t) = -R_1^{-1} B^T B_v^T S_1 x(t) \\ v_1(t) &= Bu_1(t) = -BR_1^{-1} B^T B_v^T S_1 x(t) \\ v_2(t) &= -L_2 x(t) = -R_2^{-1} B_v^T S_2 x(t) \end{aligned}$$

where S_1 and S_2 solve

$$\begin{aligned} A^T S_1 + S_1 A + Q_1 - S_1 B_u R_1^{-1} B_u^T S_1 &= 0 \\ A^T S_2 + S_2 A + Q_2 - S_2 B_v R_2^{-1} B_v^T S_2 &= 0 \end{aligned}$$

By inspection we see that

$$\begin{aligned} Q_2 &= Q_1 \\ R_2^{-1} &= BR_1^{-1}B^T \end{aligned}$$

give the same solution to the Riccati equations, $S_1 = S_2 = S$, and also the same virtual control signals, $v_1(t) = v_2(t)$.

Applying these relationships to the control law in Theorem 2 gives

$$\begin{aligned} u_2(t) &= Pv_2(t) = W^{-1}(BW^{-1})^\dagger v_2(t) \\ &= -W^{-2}B^T(BW^{-2}B^T)^{-1}BR_1^{-1}B^T B_v^T Sx(t) \end{aligned}$$

Selecting $W^2 = R_1$ yields

$$u_2(t) = -R_1^{-1} B^T B_v^T Sx(t) = u_1(t)$$

which proves that (6) achieves $u_2(t) = u_1(t)$. Note that the choice of W is not unique.

Deriving (7) is not as straightforward. To do this, we consider Design 2 but with a different control allocation objective:

$$\min_u \|\tilde{W}u\| \text{ subject to } Bu = v \tag{8}$$

From above we know that this gives the same control signal as Design 1 if

$$\begin{aligned} Q_2 &= Q_1 \\ R_2^{-1} &= BR_1^{-1}B^T \\ \tilde{W}^2 &= R_1 \end{aligned}$$

Further, (8) gives the same control law as Design 2 if

$$\tilde{W}^2 = W^2 + B^T X B$$

for any symmetric X such that $\tilde{W}^2$ is positive definite. This is true since under the constraint $Bu = v$ it holds that

$$\begin{aligned}
&\arg\min_u \|\tilde{W}u\| = \arg\min_u u^T \tilde{W}^2 u \\
&= \arg\min_u u^T (W^2 + B^T X B) u = \arg\min_u u^T W^2 u + v^T X v \\
&= \arg\min_u u^T W^2 u = \arg\min_u \|Wu\|
\end{aligned}$$

Thus, $u_1 = u_2$ is achieved for

$$\begin{aligned}
Q_1 &= Q_2 \\
R_1 &= W^2 + B^T X B
\end{aligned}$$

if there exists a symmetric matrix X that solves

$$R_2^{-1} = B R_1^{-1} B^T = B(W^2 + B^T X B)^{-1} B^T$$

and makes R_1 positive definite. We will first solve for X and then show that the resulting R_1 matrix is indeed positive definite.

Using the matrix inversion formula

$$(A + BD)^{-1} = A^{-1} - A^{-1}B(I + DA^{-1}B)^{-1}DA^{-1}$$

gives us

$$\begin{aligned}
&R_2^{-1} = B(W^2 + B^T X B)^{-1} B^T \\
&= B\Big(W^{-2} - W^{-2}B^T(I + XBW^{-2}B^T)^{-1}XBW^{-2}\Big)B^T \\
&= M - M(I + XM)^{-1}XM
\end{aligned}$$

where $M = BW^{-2}B^T$. Rearranging this expression gives

$$\begin{aligned}
XM &= (I + XM)M^{-1}(M - R_2^{-1}) \\
&= I - M^{-1}R_2^{-1} + XM - XR_2^{-1}
\end{aligned}$$

which has the solution

$$X = R_2 - M^{-1} = R_2 - (BW^{-2}B^T)^{-1}$$

Inserting this into the expression for R_1 gives

$$R_1 = W^2 + B^T\big(R_2 - (BW^{-2}B^T)^{-1}\big)B$$

What remains to show is that R_1 is positive definite. Introducing $N = BW^{-1}$ and $\tilde{u} = Wu$ we have that

$$\begin{aligned}
u^T R_1 u &= \tilde{u}^T\big(I + N^T(R_2 - (NN^T)^{-1})N\big)\tilde{u} \\
&= \tilde{u}^T\big(I + N^T R_2 N - N^T(NN^T)^{-1}N\big)\tilde{u}
\end{aligned}$$

Since N has full row rank, the singular value decomposition of N is given by

$$N = U\begin{pmatrix}\Sigma_r & 0\end{pmatrix}\begin{pmatrix}V_r^T \\ V_0^T\end{pmatrix} = U\Sigma_r V_r^T$$

where $U^T U = I$, $V_r^T V_r = I$, $V_r^T V_0 = 0$ and Σ_r is a positive definite diagonal matrix. This gives

$$N^T(NN^T)^{-1}N = V_r\Sigma_r U^T(U\Sigma_r^2 U^T)^{-1}U\Sigma_r V_r^T = V_r V_r^T$$

Parameterizing $\tilde{u}$ as $\tilde{u} = V_r\tilde{u}_r + V_0\tilde{u}_0$ now yields

$$\begin{aligned}
u^T R_1 u &= \tilde{u}_r^T\tilde{u}_r + \tilde{u}_0^T\tilde{u}_0 + \tilde{u}_r^T\Sigma_r U^T R_2 U\Sigma_r\tilde{u}_r - \tilde{u}_r^T\tilde{u}_r \\
&= \tilde{u}_0^T\tilde{u}_0 + \tilde{u}_r^T\underbrace{\Sigma_r U^T R_2 U\Sigma_r}_{\text{pos. def.}}\tilde{u}_r > 0, \quad u \neq 0
\end{aligned}$$

which shows that R_1 is indeed positive definite.

Let us finally consider the case $r \neq 0$. Since $BL_1 = L_2$ and $L_1 = PL_2$ we have that

$$\begin{aligned}
u_1(t) &= R_1^{-\frac{1}{2}}(G_{0,1}R_1^{-\frac{1}{2}})^\dagger\, r - L_1 x(t) \\
&= \tilde{W}^{-1}(G_{0,2}B\tilde{W}^{-1})^\dagger\, r - PL_2 x(t) \\
&= W^{-1}(BW^{-1})^\dagger(G_{0,2}^{-1}\, r - L_2 x(t)) = u_2(t)
\end{aligned}$$

where the last identity follows from the fact that W and $\tilde{W}$ give the same control allocation result. □

5 Discussion

Let us now discuss the implications of this "conversion theorem", relating LQ design to l_2-optimal control allocation.

The main message is that the two approaches give the designer exactly the same freedom to shape the closed loop dynamics and to distribute the control effort among the actuators. Given the design parameters of one design, Theorem 3 states how the parameters of the other design should be selected to achieve precisely the same control law.

So why then bother to split the control design into two separate tasks? Let us list some benefits of using a modular control design.

- *Facilitates tuning.* In Design 1, modifying an element of the control input weighting matrix, R_1, will affect the control distribution as well as the closed loop behavior of the system. In Design 2, the tuning of the closed loop dynamics is separated from the design of the control distribution.
- *Easy to reconfigure.* An actuator failure can often be approximately modeled as a change in the B-matrix. In Design 2 this only affects the control allocation. Hence, if the failure is detected, the new B-matrix can be used for control allocation, while the original virtual control law can still be used, provided that the damaged system is still controllable.
- *Arbitrary control allocation method.* From (2), we can see that the system dynamics are completely determined by the virtual control input, v. Hence, if we select Q_2 and R_2 as in (6), we can choose any control allocation mapping $u = h(v)$ in Design 2 such that $Bh(v) = v$, without altering the closed loop dynamics from Design 1. For a survey of control allocation methods, see, e.g., [5, 4].

- *Actuator constraints.* With a separate control allocator, actuator constraints can be handled to some extent. If the control input is bounded by $\underline{u} \leq u(t) \leq \overline{u}$, the control allocation problem in Design 2 can be reformulated as

$$\begin{aligned} u &= \arg\min_{u \in \Omega} \|Wu\| \\ \Omega &= \arg\min_{\underline{u} \leq u \leq \overline{u}} \|W_v(Bu - v)\| \end{aligned} \tag{9}$$

  Given Ω, the set of feasible control inputs that minimize $Bu - v$ (weighted by W_v), we pick the control input that minimizes u (weighted by W). This way, the control capabilities of the actuator suite can be fully exploited before the closed loop performance is degraded. Also, when $Bu = v$ is not attainable due to the constraints, W_v allows the designer to prioritize between the components of the virtual control input. The optimization problem (9) can be efficiently solved using, e.g., active set methods [8].

  Remark: It should be stressed that including the constraints in the control allocation is not equivalent to including the constraints in the original LQ problem in Design 1.

Apparently, a modular design has potential benefits. Unfortunately, not all systems with more actuators than controlled variables display the type of redundancy that can be resolved using control allocation. In some cases however, proper model approximations can be made to achieve modularity, as we will see in the design example in the following section.

6 Flight Control Example

To investigate the potential benefits of a modular LQ design we use a flight control example based on the ADMIRE model [1]. ADMIRE describes a small single engine fighter with a delta-canard configuration. To induce actuator saturations, we consider a low speed flight case, Mach 0.22, altitude 3000 m, where the control surface efficiency is poor.

The linearized aircraft model is given by

$$\begin{aligned} x &= \begin{pmatrix} \alpha & \beta & p & q & r \end{pmatrix}^T - x_{\text{lin}} \\ y &= \begin{pmatrix} \alpha & \beta & p \end{pmatrix}^T - y_{\text{lin}} \\ \delta &= \begin{pmatrix} \delta_c & \delta_{re} & \delta_{le} & \delta_r \end{pmatrix}^T - \delta_{\text{lin}} \\ u &= \begin{pmatrix} u_c & u_{re} & u_{le} & u_r \end{pmatrix}^T - u_{\text{lin}} \\ \begin{bmatrix} \dot{x} \\ \dot{\delta} \end{bmatrix} &= \begin{bmatrix} A & B_x \\ 0 & -B_\delta \end{bmatrix} \begin{bmatrix} x \\ \delta \end{bmatrix} + \begin{bmatrix} 0 \\ B_\delta \end{bmatrix} u \end{aligned} \tag{10}$$

where α = angle of attack, β = sideslip angle, p = roll rate, q = pitch rate and r = yaw rate are the aircraft state variables, δ and u contain the actual and the commanded deflections of the canard wings, the right and left elevons, and the rudder, respectively, and x_{lin}, y_{lin}, etc. are the points of linearization. All actuators have first order dynamics with a time constant of 0.05 s corresponding to $B_\delta = 20I$.

For this system, $k = m = 4$. Hence, although the number of actuators exceeds the number of controlled variables, the redundancy is not in a form that can be exploited using control allocation. Let us therefore make the two following approximations:

- The actuator dynamics are neglected, i.e., $\delta = u$ is used.
- The control surfaces are viewed as pure moment generators and their influence on $\dot{\alpha}$ and $\dot{\beta}$ is neglected. This corresponds to zeroing the top two rows of B_x.

This gives the approximate model

$$\begin{aligned} \dot{x} &= Ax + B_u u = Ax + B_v v \\ v &= Bu \end{aligned} \tag{11}$$

where $B_u = B_v B$, $B_v = \begin{bmatrix} 0_{2\times 3} & I_{3\times 3} \end{bmatrix}^T$, and B contains the last three rows of B_x. The resulting virtual control input, $v = Bu$, contains the angular accelerations in roll, pitch, and yaw produced by the control surfaces.

Let us investigate three different control strategies:

1. Standard LQ design for the approximate model (11), see Design 1, with weighting matrices Q_1, R_1.
2. LQ design and l_2-optimal control allocation for the approximate model (11), see Design 2, with Q_2 and R_2 selected as in Theorem 3. To handle actuator position constraints, the extended control allocation formulation (9) is used with $W_v = \text{diag}(1, 1, 100)$ to prioritize yaw stability.
3. Standard LQ design for the full model (10) with weighting matrices

$$Q = \begin{pmatrix} Q_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad R = R_1$$

   and L_r selected as in Theorem 1.

The weighting matrices Q_1 and R_1 are selected to achieve desirable characteristics of the short period mode, the dutch roll mode, and the roll mode.

Figure 2 shows the simulation results, based on the original linear model (10). The figure illustrates the resulting flight trajectory for each of the three designs above.

Prior to $t = 3$ s, no actuator saturation occurs. In this time interval, designs 1 and 2 above produce the exact same control signals in accordance with Theorem 3. Design 3 produces a slightly (barely visible) different result since it is based on the original, more detailed model (10).

When the roll command is applied at $t = 3$ s, the left elevons saturate. In designs 1 and 3, this causes an overshoot in the pitch variables, α and q. In design 2, the control allocator copes with the saturation by redistributing as much of the lost control effect as possible to the right elevons and to the canards. The result is that the nominal trajectory, without actuator constraints, is almost completely recovered.

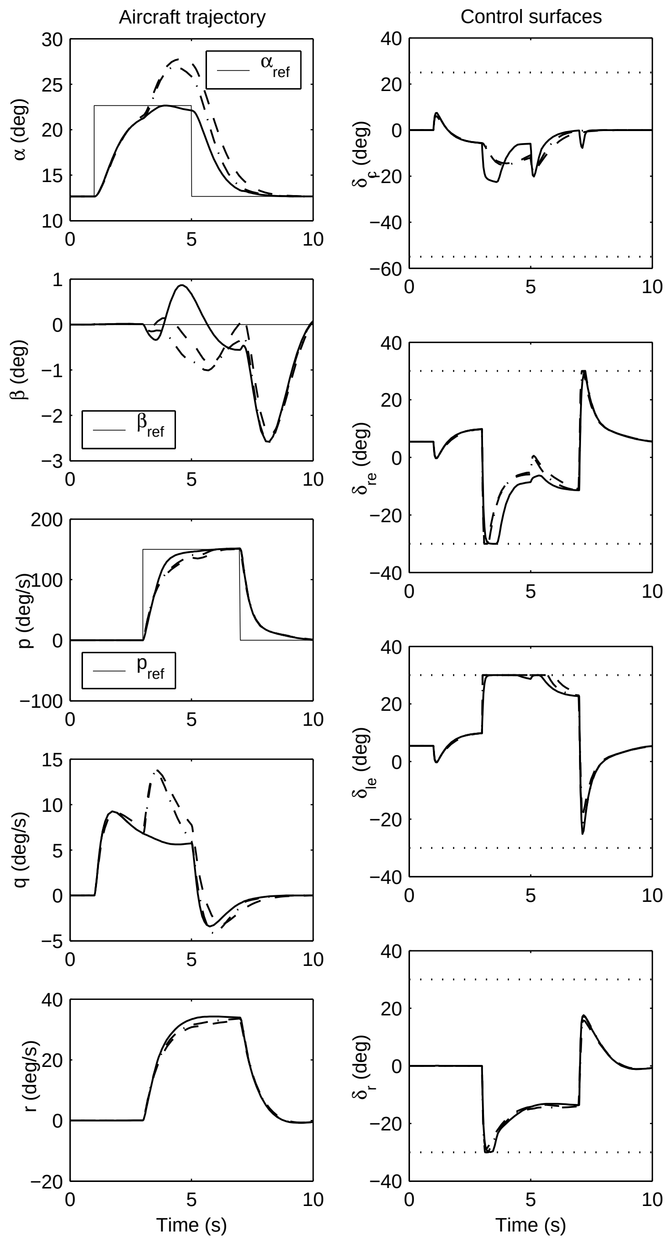

Figure 2: Aircraft trajectory, x (left), and control surface positions, δ (right), for control design 1 (dash-dotted), 2 (solid), and 3 (dashed). In design 2, the control allocator redistributes the control effort when δ_{le} saturates, preventing an overshoot in α.

7 Conclusions

For the considered class of linear systems, standard LQ design and LQ design in combination with l_2-optimal control allocation, offer the exact same design freedom in shaping the closed loop response and distributing the control effect among the actuators. Theoretically, this is an interesting result in itself since it ties together two useful tools for resolving actuator redundancy.

There are also practical implications. Given an existing LQ controller, we have shown how to split this into a new LQ controller, governing the closed loop dynamics, and a control allocator, distributing the control effect among the actuators. In the control allocator, actuator constraints can be considered, so that when one actuator saturates, the remaining actuators can be used to make up for the loss of control effect, if possible.

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